A consequence of the fact that A/I_ω is a left ideal is that we have a representation of A on this space and its completion from the left.

Definition 4.11. Let A be a unital C*-algebra and H a Hilbert space. A homomorphism of unital *-algebras $A \to BL(H,H)$ is called a *representation of A*. A representation that is injective is called *faithful*. A representation that is surjective is called *full*.

Proposition 4.12. *Let A, B be unital C*-algebras and $\phi : A \to B$ a homomorphism of unital *-algebras.*

1. $\|\phi(a)\| \leq \|a\|$ *for all $a \in A$. In particular, ϕ is continuous.*
2. *If ϕ is injective then it is isometric.*

Proof. **Exercise.** □

Theorem 4.13. *Let A be a unital C*-algebra and ω a state on A. Then, there is a natural representation $\pi_\omega : A \to BL(H_\omega, H_\omega)$. Moreover,*

$$\|\pi_\omega(a)\|^2 \geq \frac{\omega(a^*a)}{\omega(e)} \quad \forall a \in A,$$

and $\|\pi_\omega\| = 1$.

Proof. Define the linear maps $\tilde{\pi}_\omega(a) : A/I_\omega \to A/I_\omega$ by left multiplication, i.e., $\tilde{\pi}_\omega(a) : [b] \mapsto [ab]$. That $\tilde{\pi}_\omega(a)$ is well defined follows from Proposition 4.9 (I_ω is a left ideal). By definition we have then $\tilde{\pi}_\omega(ab) = \tilde{\pi}_\omega(a) \circ \tilde{\pi}_\omega(b)$ and $\tilde{\pi}_\omega(e) = \mathrm{id}_{A/I_\omega}$. Furthermore, $\|\tilde{\pi}_\omega(a)\| \leq \|a\|$ due to Proposition 4.8.3 and hence $\tilde{\pi}_\omega(a)$ is continuous. So we have a homomorphism of unital algebras $\tilde{\pi}_\omega : A \to BL(A/I_\omega, A/I_\omega)$. Also, $\tilde{\pi}_\omega$ preserves the *-structure because,

$$\langle \tilde{\pi}_\omega(a^*)[b], [c] \rangle_\omega = [a^*b, c]_\omega = \omega(c^*a^*b) = [b, ac]_\omega = \langle [b], \tilde{\pi}_\omega(a)[c] \rangle_\omega.$$

Since $\tilde{\pi}_\omega(a)$ is continuous it extends to a continuous operator $\pi_\omega(a) : H_\omega \to H_\omega$ on the completion H_ω of A/I_ω, with the same properties. In particular, π_ω is a homomorphism of unital *-algebras.

Due to the bound $\|\tilde{\pi}_\omega(a)\| \leq \|a\|$ and hence $\|\pi_\omega(a)\| \leq \|a\|$ (or due to Proposition 4.12.1) we find $\|\pi_\omega\| \leq 1$. Observe also that $\omega(e) \neq 0$ (since otherwise $\omega = 0$ by Proposition 4.8.2 or 4.8.3) and hence $\|\pi_\omega(a)\|^2 \geq [ae, ae]_\omega / [e, e]_\omega = \omega(a^*a)/\omega(e)$. In particular, $\|\pi_\omega\| \geq \|\pi_\omega(e)\| \geq 1$. Thus, $\|\pi_\omega\| = 1$. □

The construction leading to the Hilbert spaces H_ω and this representation is called the *GNS-construction* (Gelfand-Naimark-Segal).

Definition 4.14. Let A be a unital C*-algebra, H a Hilbert space and $\phi : A \to BL(H,H)$ a representation. A vector $\psi \in H$ is called a *cyclic vector* iff $\{\phi(a)\psi : a \in A\}$ is dense in H. The representation is then called a *cyclic representation*.

Proposition 4.15. *Let A be a unital C^*-algebra and ω a state on A. Then, there is a cyclic vector $\psi \in H_\omega$ with the property $\omega(a) = \langle \pi_\omega(a)\psi, \psi\rangle_\omega$ for all $a \in A$.*

Proof. **Exercise.** □

A deficiency of the representation of Theorem 4.13 is that it is neither faithful nor full in general. Lack of faithfulness can be remedied. The idea is that we take the direct sum of the representations π_ω for all normalized states ω.

Proposition 4.16. *Let $\{H_\alpha\}_{\alpha\in I}$ be a family of Hilbert spaces. Consider collections ψ of elements $\psi_\alpha \in H_\alpha$ with $\alpha \in I$ such that $\sum_{\alpha\in J} \|\psi_\alpha\|^2 < \infty$ for all finite subsets $J \subseteq I$. Then, the set H of such collections ψ is naturally a Hilbert space and we have isometric embeddings $H_\alpha \to H$ for all $\alpha \in I$.*

Proof. **Exercise.** □

Definition 4.17. The Hilbert space H constructed in the preceding Proposition is called the *direct sum* of the Hilbert spaces H_α and is denoted $\bigoplus_{\alpha\in I} H_\alpha$.

Proposition 4.18. *Let A be a unital C^*-algebra, $\{H_\alpha\}_{\alpha\in I}$ a family of Hilbert spaces and $\phi_\alpha : A \to BL(H_\alpha, H_\alpha)$ a representation for each $\alpha \in I$. Then, there exists a representation $\phi : A \to BL(H, H)$ such that $\|\phi(a)\| = \sup_{\alpha\in I} \|\phi_\alpha(a)\|$ for all $a \in A$, where $H := \bigoplus_{\alpha\in I} H_\alpha$.*

Proof. **Exercise.** □

At this point we need a few more facts about states.

Proposition 4.19. *Let A be a unital C^*-algebra and $\omega : A \to \mathbb{C}$ linear. Then, ω is a state iff ω is continuous and $\|\omega\| = \omega(e)$.*

Proof. **Exercise.** □

Proposition 4.20. *Let A be a unital C^*-algebra. Then, for each $a \in A$ positive there exists a normalized state ω such that $\omega(a) = \|a\|$.*

Proof. **Exercise.** □

We are now ready to put everything together.

Theorem 4.21 (Gelfand-Naimark)**.** *Let A be a unital C^*-algebra. Then, there exists a Hilbert space H and a faithful representation $\pi : A \to BL(H, H)$.*

Proof. **Exercise.** □

This result concludes our characterization of the structure of C*-algebras: Each C*-algebra arises as a C*-subalgebra of the algebra of bounded operators on some Hilbert space.